Practice Exam 2 Solutions

1.) let G be an Abelian grp s.t. $|G|=16$. Suppose $\exists\, a,b\in G$ s.t. $|a|=|b|=4$ and $a^2\neq b^2$. Determine the isomorphism class of G.

pf:

Since $|G|=2^4$ and G Abelian, by the Fundamental Theorem of finite Abelian grps we have $G\cong \mathbb{Z}_{16}$, $G\cong \mathbb{Z}_8\oplus\mathbb{Z}_2$, $G\cong\mathbb{Z}_4\oplus\mathbb{Z}_4$, $G\cong\mathbb{Z}_4\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2$, or $G\cong\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2$

1st notice in $\mathbb{Z}_n$ a^2 is really $2a$ as $a^2=a\cdot a$ but addition is the op. so get $2a$.

Then in $\mathbb{Z}_{16}$ $|4|=|12|=4$ but $2\cdot 4=8$ and $2\cdot 12=24\equiv_{16} 8$ so $2(4)=2(12)$ so $G\not\cong\mathbb{Z}_{16}$

In $\mathbb{Z}_8\oplus\mathbb{Z}_2$ $|(2,1)|=|(6,1)|=4$ but $2(2,1)=2(6,1)=(4,1)$ so $G\not\cong\mathbb{Z}_8\oplus\mathbb{Z}_2$

In $\mathbb{Z}_4\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2$ $|(1,1,1)|=|(3,1,1)|=4$ but $2(1,1,1)=2(3,1,1)=(2,2,2)$ so $G\not\cong\mathbb{Z}_4\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2$

Finally $\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2$ has no elements of order 4 so $G\cong\mathbb{Z}_4\oplus\mathbb{Z}_4$ ☐

2) let $p_1,\dots,p_n$ be distinct primes and G Abelian grp s.t. $|G|=p_1\cdots p_n$. What is G isomorphic to?

pf:

By the Fundamental Theorem of finite Abelian grps, $G\cong\mathbb{Z}_{p_1\cdots p_n}$ or $G\cong\mathbb{Z}_{p_1}\oplus\mathbb{Z}_{p_2\cdots p_n}$, or, ... $G\cong\mathbb{Z}_{p_1}\oplus\mathbb{Z}_{p_2}\oplus\cdots\oplus\mathbb{Z}_{p_n}$

Recall a corollary said ~~[illegible]~~ $\mathbb{Z}_{pq}\cong\mathbb{Z}_p\oplus\mathbb{Z}_q$ iff $\gcd(p,q)=1$. Using an inductive argument on n, with this corollary shows $G\cong\mathbb{Z}_{p_1p_2\cdots p_n}$. ☐

3.) let $S=\{a+bi : a,b\in\mathbb{Z},\ b \text{ even}\}$. Show S is a subring, but not an ideal in $\mathbb{Z}[i]$.

pf:

pick $a_1+b_1i,\ a_2+b_2i\in S$, then $b_1=2k_1$ and $b_2=2k_2$ for $k_1,k_2\in\mathbb{Z}$. then $a_1+b_1i-(a_2+b_2i)=$
$=(a_1-a_2)+i(b_1-b_2)$ but $b_1-b_2=2(k_1-k_2)$ so b_1-b_2 is still even. next.

$(a_1+b_1i)(a_2+b_2i)=a_1a_2-b_1b_2+i(b_1a_2+a_1b_2)$ but $b_1a_2+a_1b_2=2(k_1a_2+a_1k_2)$ which is still even.

So S is a subring of $\mathbb{Z}[i]$. Now consider $1+3i\in\mathbb{Z}[i]$, and $1+2i\in S$ and $(1+3i)(1+2i)=1-6+i(3+2)=-5+5i$ but 5 is not even so $(1+3i)(1+2i)\notin S$ so S is not an ideal of $\mathbb{Z}[i]$. ☐

4.) let R be a commutative ring, A,B ideals of R s.t. $A=\langle a\rangle$, $B=\langle b\rangle$. Define $I=$ ~~[illegible]~~ $\langle ab\rangle=\{r(ab):r\in R\}$. Prove I is an ideal of R.

pf:

since I is just a set we need to show $x-y\in I$ for $x,y\in I$ and $rx, xr\in I$ for $x\in I$ and $r\in R$.

pf:

let $x,y\in I$. ~~[illegible]~~ then $x=r_1(ab)$ and $y=r_2(ab)$ so $x-y=r_1(ab)-r_2(ab)$
$=(r_1-r_2)(ab)\in I$ since $r_1-r_2\in R$. Then $rx=r(r_1(ab))=(rr_1)(ab)\in I$ and
$xr=r_1(ab)r=(r_1r)(ab)\in I$ since R is commutative so $rx, xr\in I\Rightarrow I$ is an ideal. ☐

5.) Show $\mathbb{R}[x]/\langle x^2+1\rangle$ is a field.

pf:
1st note $\mathbb{R}[x]/\langle x^2+1\rangle = \{p(x)+\langle x^2+1\rangle : p(x)\in\mathbb{R}[x]\}$ but notice $x^2+1+\langle x^2+1\rangle = 0+\langle x^2+1\rangle$
so in $\mathbb{R}[x]/\langle x^2+1\rangle$, $0=x^2+1 \Rightarrow x^2=-1$ moreover by division algorithm $p(x)=g(x)(x^2+1)+r(x)$
w/ degree of $r(x)=0$ or 1 so in $\mathbb{R}[x]/\langle x^2+1\rangle \Rightarrow p(x)=r(x)$ and $r(x)=ax+b$ for $a,b\in\mathbb{R}$
$\Rightarrow$ all elements are of the form $ax+b+\langle x^2+1\rangle$ in $\mathbb{R}[x]/\langle x^2+1\rangle$. Then since $x^2=-1$ in the factor ring $\Rightarrow x=\pm i$ so consider the ring homom. $\varphi: \mathbb{R}[x]/\langle x^2+1\rangle \to \mathbb{C}$ via $\varphi(ax+b)=ai+b$
Then it's clear to check φ is 1-1, onto and op. pres. so φ is isom. and $\mathbb{C}$ is field so $\mathbb{R}[x]/\langle x^2+1\rangle$ must be a field Δ.

6.) let $I=\langle 3+i\rangle$, known I is ideal of $\mathbb{Z}[i]$. Prove or disprove I is prime.

pf:
consider the quotient ring $\mathbb{Z}[i]/I = \{a+bi+\langle 3+i\rangle : a,b\in\mathbb{Z}\}$. But $3+i+I=0+I$ so in $\mathbb{Z}[i]/I$, $3+i=0$ or $i=-3 \Rightarrow -1=9$ or $0=10$ so $\mathbb{Z}[i]/I=\{a+I : a=0,\dots,9\}$
so clear $0+I$ is additive id, and $1+I$ is the unity, so $10(1+I)=10+I=I$ so $|1+I|$ is at most 10
but $|1+I|=1,2,5$ or 10, if $|1+I|=5 \Rightarrow 5(1+I)=0+I$ or $5\in I \Rightarrow \exists\, a,b\in\mathbb{Z}$ s.t. $(3+i)(a+bi)=5$
$\Rightarrow \begin{cases}3a-b=5\\a+3b=0\end{cases}$ but this has no integer sols so $|1+I|\neq 5$ if $|1+I|=2 \Rightarrow \begin{cases}3a-b=2\\a+3b=0\end{cases}$ again no integer sols
similarly for if $|1+I|=1$ so $|1+I|=10 \Rightarrow \mathbb{Z}[i]/I$ has 10 elements so $\mathbb{Z}[i]/I\cong\mathbb{Z}_{10}$
but $\mathbb{Z}_{10}$ is not an integral domain as $2\cdot 5=10=0$ and $2,5\neq 0$ so $\mathbb{Z}[i]/I$ not int domain
$\Rightarrow I$ is not prime ideal. Moreover it's not maximal either as $\mathbb{Z}_{10}$ can't be a field. ⊠

7.) let $R=\left\{A=\begin{pmatrix}a&b\\0&c\end{pmatrix} : a,b,c\in\mathbb{Z}\right\}$ let $\varphi:R\to\mathbb{Z}$ s.t. $\varphi(A)=a$. Show φ is ring homom. $\ker\varphi=?$

pf:
let $A_1=\begin{pmatrix}a_1&b_1\\0&c_1\end{pmatrix}$, $A_2=\begin{pmatrix}a_2&b_2\\0&c_2\end{pmatrix}$ then $A_1+A_2=\begin{pmatrix}a_1+a_2&b_1+b_2\\0&c_1+c_2\end{pmatrix}$ and $\varphi(A_1+A_2)=a_1+a_2=\varphi(A_1)+\varphi(A_2)$
next $A_1A_2=\begin{pmatrix}a_1&b_1\\0&c_1\end{pmatrix}\begin{pmatrix}a_2&b_2\\0&c_2\end{pmatrix}=\begin{pmatrix}a_1a_2&a_1b_2+b_1c_2\\0&c_1c_2\end{pmatrix}$ so $\varphi(A_1A_2)=a_1a_2=\varphi(A_1)\varphi(A_2)$
so φ is a ring homom. next $\ker\varphi=\{A:\varphi(A)=0\}$ so $\varphi(A)=0$ and $\varphi(A)=a \Rightarrow a=0$
so $\ker\varphi=\left\{\begin{pmatrix}0&b\\0&c\end{pmatrix} : b,c\in\mathbb{Z}\right\}=\left\langle\begin{pmatrix}0&1\\0&1\end{pmatrix}\right\rangle$ ⊠

8.) let R be a comm. ring s.t. char $R=p$, prime. show $\varphi:R\to R$ $\varphi(x)=x^p$ is ring homom.

pf:
1st let $x,y\in R$, then $\varphi(xy)=(xy)^p=x^py^p=\varphi(x)\varphi(y)$ since R is commutative.
then $\varphi(x+y)=(x+y)^p=\sum_{k=0}^{p}\binom{p}{k}x^{p-k}y^k=\binom{p}{p}x^p+\binom{p}{p-1}x^{p-1}y+\dots+\binom{p}{0}y^p$.
$=x^p+y^p+\binom{p}{p-1}x^{p-1}y+\dots+\binom{p}{1}xy^{p-1}$ since char $R=p$ and $\binom{p}{k}$ for $k=1,\dots,p-1$ will always have a coefficient w/ p, these terms are zero. so $\varphi(x+y)=x^p+y^p=\varphi(x)+\varphi(y)$.
so φ is ring homom. ⊠

9.) let $f(x) \in \mathbb{R}[x]$. Suppose $f(a)=0$ and $f'(a)=0$. show $(x-a)^2 \mid f(x)$.

pf ~

1^{st} since $f(a)=0$. Then by the division algorithm $(x-a)$ is a factor of $f(x)$ and $f(x)=(x-a)q(x)$ so now since $f'(a)=0 \Rightarrow (x-a)$ is a factor of $f'(a)$ via the div. algorithm also. but $f'(x)=q(x)+(x-a)q'(x)$ but $f'(a)=q(a)$ $\Rightarrow$ a is root of q so $q(x)=(x-a)g(x)$ $\Rightarrow f(x)=(x-a)q(x)=(x-a)(x-a)g(x)=(x-a)^2 g(x)$ $\Rightarrow (x-a)^2 \mid f(x)$ □